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Shock Wave Generation and Cut off Condition in Nonlinear Series Connected Discrete Josephson Transmission Line

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***Abstract*— The nonlinear wave propagation in a series-connected Discrete Josephson Junction Transmission Line (DJTL) is investigated. This structure consists of a superconductive Coplanar Waveguide (CPW), that is periodically loaded by either single or lumped arrays of Josephson junctions (JJs). Each junction is represented by the basic circuit model which leads to a nonlinear inductor element. Having a significant number of junctions per wavelength, the discrete transmission line (TL) can be considered as a uniform nonlinear transmission line. The nonlinear wave equations are solved numerically by Finite Difference Time Domain (FDTD) method. Features and characteristics such as cut-off propagation, dispersive behavior and shock wave formation, which are expected from wave propagation through the nonlinear DJTL, are discussed.**

***Index Terms*— Applied superconductivity, Microwave superconductivity, Discrete Josephson transmission line, Finite Difference Time Domain Method, Nonlinear microwave propagation, Josephson junction devices, Nonlinear transmission lines, Shock waves, Dispersion.**

I. Introduction

With the discovery of supercurrent tunneling through the Josephson junction, physicists and engineers have been able to develop several new devices that have extraordinary characteristics [*1*]. This has broadened the horizon of RF and microwave theory and applications, and it retains to this day a matured class of knowledge and technology which is referred to as Microwave Superconducting Electronics [*2*]. The unique nonlinear property of Josephson junction is the main motivation to design and fulfill structures and devices such as nonlinear transmission lines [*3*], soliton propagators [*4*], microwave oscillators [*5*], mixers, detectors [*6*], [*7*], parametric amplifiers [*8*] , SQUID amplifier [*9*], digital information processing, and analog amplifier and transistor [*4*]. These devices can be used in microwave systems, electronic components [*2*], superconducting optoelectronics [*10*] and quantum computer circuitry [*11*].

The continuous long Josephson transmission line (JTL) shows very promising attributes [*4*], but the discrete counterpart (DJTL) has demonstrated to have superior properties for microwave applications and electronics due to its microwave compatibility, impedance matching, uniform distribution of biasing current, higher wave velocity and low cut-off frequency. Wave propagation through parallel-connected DJTL has been studied in [*3*], because it was confidently believed to acquire very similar characteristics as those in continuum JTL. By circuit analysis in JJSPICE solver, shock wave formation in a series connected DJTL has already been reported for very low frequencies [*12*].

In this paper, a practical series-connected DJTL implemented on a 50Ω CPW is proposed. We analyze the structure in section III with a systematic and robust approach to capture the wave nature of the traveling wave through the structure. In section IV, we analytically study the cut off condition of the DJTL. In part V, we develop a stable and rigorous FDTD technique to solve the nonlinear wave equation, followed by the simulation results presented in section V.

II. Series-Connected DJTL on CPW

To construct a series DJTL, the structure of Fig 1 (a) is proposed. This is a $Z_0 = 50\Omega$ superconducting coplanar waveguide (CPW) which is periodically loaded by an array of Josephson junctions. We use Al-Al_2O_3-Al junctions as they have small critical current and small junction capacitance, typically $I_c = 1-2\mu\text{A}$ and $C_J = 5\text{fF}$ [*13*]. Therefore, by substituting critical current into the equation $L_{J0} = \Phi_0/2\pi I_c$, the series associated inductance to the single junction will be relatively large, $L_{J0} = 0.17\text{nH}$, where Φ_0 is the magnetic flux quanta with value of $\Phi_0 = 2.0679\times10^{-15}\ \text{T.m}^2$. Moreover, by lowering the temperature far below the critical temperature of Al, the normal resistive channel of the junction can be removed, and the basic model of the JJ can represent the electrical performance of each junction, provided the following current is restricted less than I_c. By utilizing this model each junction can be replaced by a nonlinear inductor satisfying

$$L_J(I) = \frac{L_{J0}}{\sqrt{1-\left(\frac{I}{I_c}\right)^2}} \qquad (1)$$

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where I is the current flow passing through the junction. The distributed inductance and capacitance of a 50Ω TEM-transmission line are $L = 166\ \mathrm{nH/m}$ and $C = 66\ \mathrm{pF/m}$, so by putting an array of $N = 10$ junctions at each block and distributing these blocks at every centimeter of the line, the total amount of nonlinear distributed JJ inductance will be comparable to the distributed inductance of the line.

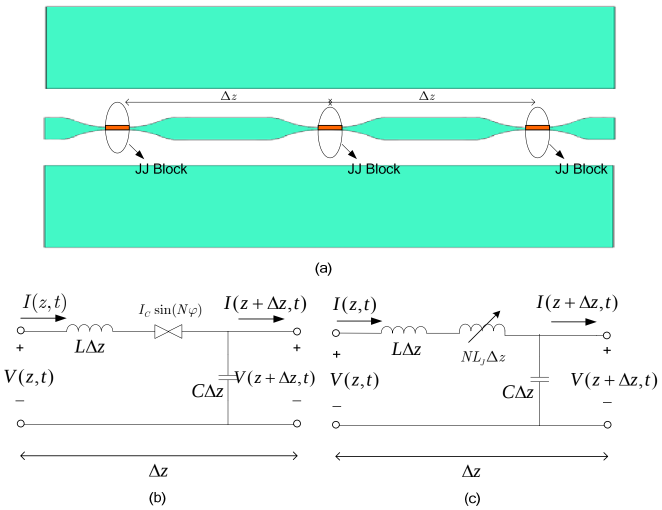

Fig 1. (a)Realization of series-connected DJTL on a CPW. (b), (c) Distributed circuit model of series-connected DJTL with basic JJ and nonlinear inductor, respectively, for an array of N junctions. The period of the transmission line is denoted by Δz.

III. Transmission Line Equations for series DJTL

If number of JJ blocks at each wavelength is roughly more than 20 ($\Delta z \leq \lambda/20$), they are distributed uniformly along the waveguide, so slow-varying approximation can be applied to the structure. Therefore, the structure can be considered effectively homogenous [*14*] and discrete inductance associated to each Josephson block can be stated in the unit of $\mathrm{nH/m}$. The transmission line model for this effectively homogeneous nonlinear structure is illustrated in Fig 1(c). The nonlinear equations that characterize the propagation of the voltage and current are

$$\begin{cases} \dfrac{\partial V}{\partial z} = -L\dfrac{\partial I}{\partial t} - NI_c L_{J0}\dfrac{\partial}{\partial t}\left[\sin^{-1}\left(\dfrac{I}{I_c}\right)\right] \\ \dfrac{\partial I}{\partial z} = -C\dfrac{\partial V}{\partial t} \end{cases}. \tag{2}$$

As mentioned in the previous section, instead of a single junction, there exists an array of N junctions at each unit cell, such that the effective mounted inductance will be NL_{J0} instead of L_{J0}, as illustrated in Fig 1 (c). Eliminating one of these two coupled variable, i.e. voltage, results in the following nonlinear wave equations for the current

$$\frac{\partial^2 I}{\partial z^2} = LC\frac{\partial^2 I}{\partial t^2} + NL_{J0}CI_c\frac{\partial^2}{\partial t^2}\left[\sin^{-1}\left(\frac{I}{I_c}\right)\right]. \tag{3}$$

Putting a voltage source and a load impedance at the ends of the DJTL and turning the DJTL on at $t = 0$, necessary boundary and initial conditions are provided in the form of

$$V_s(t) = R_s I(0,t) + V(0,t) \tag{4}$$

$$V(z_{max},t) = R_L I(z_{max},t) \tag{5}$$

Equations (4) and (5) are necessary to have a unique wave solution. In above equations, $V_s(t)$ is the waveform of the voltage source, R_s is the associated series resistance with the source, and R_L is the load resistor. As the boundary conditions at the ends of the structure involve both current and voltage variables, they are called mixed-boundary conditions. Therefore, the coupled equations in (2), seems more suitable than the single (3) to address the physical behavior of the DJTL.

Equation (3) can be linearized by applying small signal approximation ($I \ll I_c$), so we expand the inverse sine function in Traylor's series, and then we hold the first term and ignore all other terms, i.e. $\sin^{-1}(I/I_c) \approx I/I_c$. Inserting a solution in the form of the plane wave harmonic, $I = Re\{I_0 e^{j(\omega t - kz)}\}$ into the linearized equation, results in the following dispersion relation

$$k^2 = (L + NL_{J0})C\omega^2 \tag{6}$$

The constant I_0 is the complex amplitude of the plane wave, and k is the phase constant of the wave.

IV. Ladder Network Equations for series DJTL

DJTL can also be modeled by a ladder network. This has been developed for the parallel-connected DJTL, and the resultant discrete equations which resemble the evolution of flux (φ) in continous long Josephson junction is usually referred to as fluxon dynamic equations [*15*]. In this view, DJTL is divided into N identical segments; each covers one period. Each segment (or period) is displayed by an LC circuit connected to the JJ block, as shown in Fig 2, and discrete index n accommodates the continuous variable z. As the transmission line part of each period is modeled by a single LC circuit, this model is also valid in the low frequency domain, when the length of transmission line (period Δz) is small compared to the wavelength. However, this model captures the effect of discreteness of the line, as explained at the end of this section.

Similar to the parallel-connected DJTL, we attain the following equation to express flux propagation in this structure

$$\frac{1}{\Delta z^2}(\sin\varphi_{n-1} - 2\sin\varphi_n + \sin\varphi_{n+1}) = NL_{J0}C\frac{d^2\varphi_n}{dt^2} + LC\frac{d^2}{dt^2}[\sin\varphi_n] \tag{7}$$

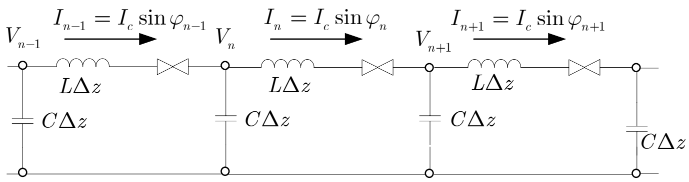

Fig 2. The discrete circuit model of DJTL

where $1 \leq n \leq N$ and φ_n is the total superconducting phase difference on the nth junction block as shown in Fig 2. Therefore, there are N nonlinear ordinary differential equations, all in the format of (7). Assigning proper boundary and initial conditions, the system of differential equations must be solved numerically. Afterward, by knowing all φ_n at each unit cell, we are able to find the associated voltage and current to the unit cell. Considering a particular harmonic solution of $\varphi_n = Re\{\varphi_0 e^{j(\omega t - \kappa n)}\}$ with constant and small amplitude approximation ($\sin\varphi_n \sim \varphi_n$) and substituting this into the discrete equation (7), this yields the following dispersion relation

$$\sin^2(\kappa/2) = \frac{1}{4}\omega^2 C\left(L + NL_{J0}\right)\Delta z^2. \tag{8}$$

Parameter κ stands for the phase difference between two subsequent cells. If the phase constant and period of the structure is k and Δz, parameter κ will be equal to $k\Delta z$. If the long wave approximation ($\Delta z/\lambda \to 0$) holds, $\kappa = k\Delta z$ is very small and by using the approximation of $2\sin^2(\kappa/2) \sim \kappa^2/2 = k^2\Delta z^2/2$ two dispersion relations in (6) and (8) match very well. However, as the frequency approaches the Bragg cut-off frequency, the wave becomes dispersive and these two dispersion relations deviate from each other. Based on (8), to have a wave propagation condition, the following condition must be met

$$\omega \leq \frac{2}{\Delta z\sqrt{C\left(L + NL_{J0}\right)}}. \tag{9}$$

The right hand side of this inequality is the Bragg cut-off frequency which corresponds to the phase constant of $k = \pi/\Delta z$. The cut-off behavior of DJTL has been formerly mentioned in [*6*],[*16*].

V. Finite Difference Time Domain Method

By driving DJTL below the Bragg frequency, the transmission line model is accurate enough to describe the behavior of the traveling wave through the DJTL. As the wave equation for DJTL structure is a set of nonlinear partial differential equations in time and space domain, we developed a rigorous FDTD solver based on the Lax-Wendroff explicit scheme to solve the wave equations [*17*]. The validity of this tool has been justified by comparing the results to those produced by explicit Crank-Nicolson technique. The detail of this FDTD technique is described in [*18*]. With this method, one can monitor the time evolution of an incoming signal with any shape such as sinusoidal and Gaussian, during its trip along the DJTL and its interaction with other signals. Also, transient, steady state response of the line, and shape forming/deforming of any pulse can be investigated.

VI. Numerical Results

The physical parameters of the DJTL under study are chosen as those mentioned in section II. i.e. $L = 166$ nH/m, $C = 66$ pF/m, $I_c = 2\mu A$, $L_{J0} = 0.17$nH and the number of junctions at each block is $N = 10$. Assuming the period of $\Delta z = 1$cm, the effective distributed JJ inductance can be expressed in the unit nH/m by expression $NL_{J0}/\Delta z$ which yields 170 nH/m. Bragg cut off frequency can be found by using the right hand side of the inequality (9), $f_B = 6.75$ GHz. A sinusoidal voltage source drives the DJTL with amplitude of $V_s = 10\mu$V, frequency of $f_s = 1.2$ GHz and matched series impedance of $R_s = 50\Omega$. The line is ended with a load impedance of $R_L = 50\Omega$. Meeting the condition $\omega_s < \omega_B$, wave propagates through the DJTL. Fig 3 shows the voltage wave propagation over time and space axes.

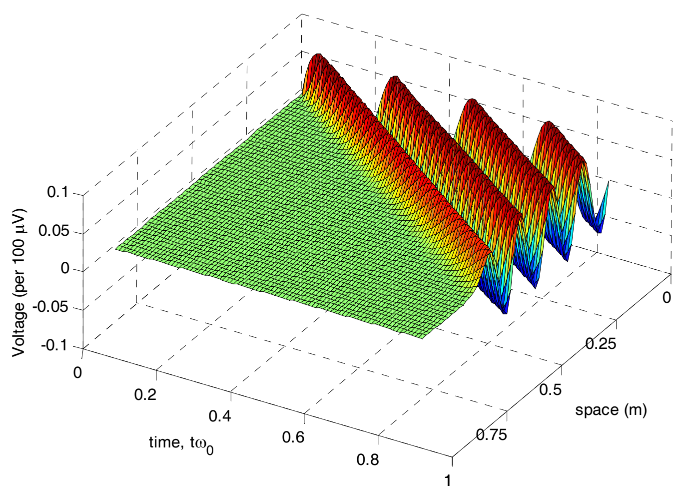

Fig 3. Propgation of sinosoideal wave in a DJTL. . $L = 166$ nH/m, $C = 66$ pF/m, $I_c = 2\mu$A, $L_{J0} = 0.17$nH, $N = 10$, $\Delta z = 1$cm, $V_s = 10\mu$V, $f_s = 1.2$ GHz, $R_s = 50\Omega$, $R_L = 50\Omega$.

To justify the validity of the FDTD code, we remove the JJ block by letting $L_{J0} = 0$, put a 50Ω matched load at the end, and drive the DJTL by the matched source at frequency $f_s = 1.2$ GHz. The regular response of linear transmission line is expected, with no reflection at the end. Also, the amplitude of the voltage wave traveling through the waveguide must be one half of the amplitude of the voltage source. These are in excellent agreement with the result in Fig 4.

By changing the physical parameters of the DJTL, the cut off frequency will change. For example, by putting $N = 300$ junctions at each JJ block and increasing the period to $\Delta z = 2$cm, the cut off frequency is found as $f_B = 1.18$GHz. Hence, if the driving voltage sources has the frequency of $f_s = 1.2$ GHz, inequality (9) fails to satisfy and the cut off conduction occurs as seen in Fig 5.

According to equation (1), nonlinear Josephson inductance grows as current increases, so we expect that high-current sections of the waveform to propagate slower than the low-current sections. This property of series-connected DJTL cause shock wave formation as displayed in Fig 6 for the Gaussian pulse.

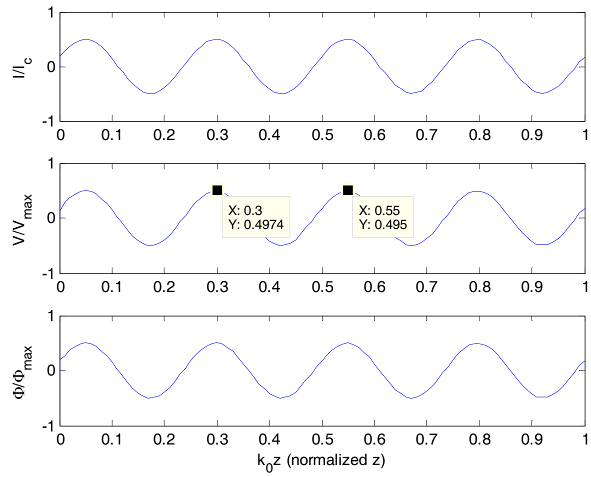

Fig 4. The wave pattern of a regular transmission line connected to the matched source and load with L = 166 nH/m, C = 66 pF/m, L_{J0} = 0 nH.

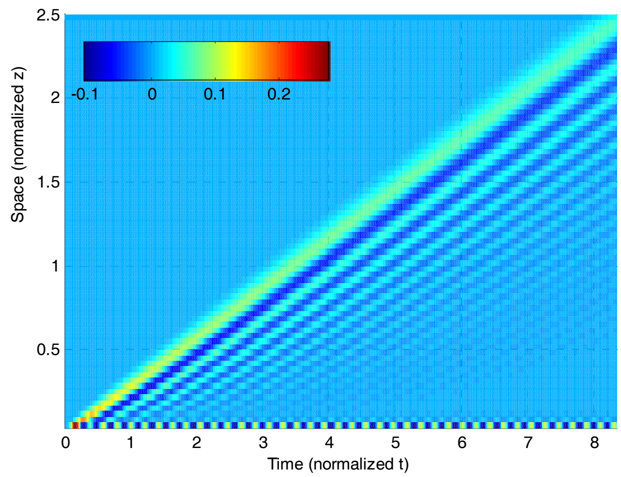

Fig 5. Stopped-propagation of voltage wave through a DJTL, L = 166 nH/m, C = 66 pF/m, L_{J0} = 0.17 nH, I_c = 2 μA, N=300, Δz = 2 cm, V_s = 20 μV, f_s = 1.2 GHz, R_s = 50 Ω, R_L = 50 Ω.

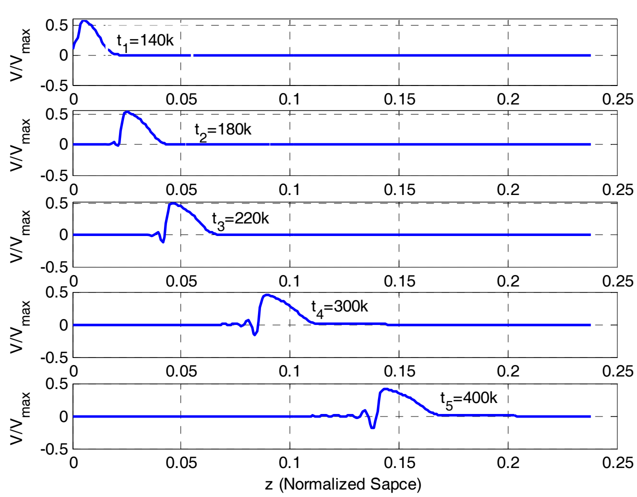

Fig 6. Sketch of the formation of a shock wave in a nonlinear Josephson junction transmission line.

VII. CONCLUSION

By periodically loading a superconducting CPW with discrete blocks containing an array of overdamped [6] Josephson junctions, a series-connected DJTL is proposed to realize a nonlinear TL. Instead of the conventional approach of circuit theory, the transmission line analysis is invoked to explore the electromagnetic propagation in the structure. The nonlinear wave equations are solved by FDTD method based on the explicit Lax-Wendroff scheme. Shock wave formation and cut off propagation are demonstrated in simulation results.

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